

Fawly Powers⁰

(A Play in Three Acts)

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Act 1.

Scene: The lobby at the Fawly Towers Hotel in Torquay. Basil is hunched over the typewriter behind the reception desk busily typing away with one finger preparing the lunch menu. Sybil has just come downstairs and is fast approaching....

Sybil Ah! There you are Basil, haven't you finished that lunch menu yet? You'd better get a move on or it'll soon be teatime.

Basil: (startled, pauses, frowns, looks round absently) Mmh. Yes dear? Nearly done dear, light of my life. Don't know why we bother with a luncheon menu really, the kind of ignorant rabble we get in here. Most of them look as if they'd be more at home with a large trough of baked beans and a small shovel. I'm quite busy actually dear. Have *you* been busy? Polishing your nails again? Having a nice lie down perhaps? Be sure not to overdo it won't you?

Sybil: No dear, actually I've been upstairs talking to that nice young couple in Number 7. They're staying for two weeks and they'll be taking some evening meals in their room. They mustn't be disturbed because the husband is writing up his dissertation on "Mathematics in Media Studies" and he wants peace and quiet to concentrate.

Basil: (snorts) Yes dear. I already know dear, thank you very much. I spoke to your young Einstein this morning and I can't say that I was impressed. In fact he didn't seem quite all there to me. If you stood close enough to him and listened carefully you'd probably be able to hear the sea. The silly ass had the nerve to ask me where he could buy a mouse! I mean, really! What does he think this is, a pet shop?! I left him in no doubt that rats, mice, Siberian hamsters and all other forms of vermin are strictly against health regulations and definitely NOT allowed in this hotel¹. He seemed to find this amusing somehow judging by the vacuous grin he gave me. Then he tried to tell me I should have put some windows in my typewriter! What strange people we seem to get in here. He's clearly depriving a village somewhere of an idiot. If you give him a penny for his thoughts expect change. We'd better keep an eye on him all the same. If he starts asking for extra cheese with those room meals let me know at once.

Sybil: Basil! That *vacuous grin* of his was obviously one of embarrassment and pity for your pathetic ignorance. The '*mouse*' he referred to will be one for his computer, and he was only pulling your leg about *Microsoft Windows*, don't you know anything at all? Our little hotel clearly already has its full complement of idiots.

Basil (looks quizzical and uncertain for a second before recovering his petulance): Anyway what exactly is "Mathematics in Media Studies" that's what I'd like to know? And just where *is* this University of Lower Standards I wonder? I've never heard of it. Come to think of it though it could be almost anywhere. Anybody can get a degree these days if they can only borrow enough money. All they have to do is hand over enough cash to one of those holiday camps they call 'New Universities' and hang around the bar for three years. It wasn't like that in my day I can tell you, Oh No! I should cocoa, no molly-coddling then. Anyway I mustn't waste any more of my valuable time with idle gossip, I've got more important work to do. (*pointedly turns his back and continues typing away with one finger*)

Sybil: Lower Stanstead! Basil, It's Lower Stanstead not Lower Standards! He's at the University of Lower Stanstead. Is there something wrong with your ears dear? Anyway all that's as maybe but they do seem to be rolling in it and the wife's a Doctor too. Charming couple they are.

Basil: (stops typing again) Doctor eh? And loads of dosh to boot eh? Well er uhm of course it's just possible my initial assessment may perhaps have been a smidgeon on the hasty side. Will they be paying in advance did you ask dear, thereby availing themselves of our recently introduced and, in my opinion, over-generous 0.25 % discount scheme?

Sybil: They already have dear, and she paid up with a Titanium card. There is just one small thing though. We got chatting about this and that, you know how it is, and I happened to mention about your having a degree in mathematics and all those University prizes and medals you won before I met you. How you managed that has always mystified me. I told them you might be willing to help out with the mathematics if they should get stuck writing up and as it happens they jumped at it. In fact they've already come up with a teensy-weensy maths problem for you to have a look at.

Basil: (looking aghast) You did WHAT you silly woman?! You may not have ulcers yourself but you're definitely a carrier! Isn't it blatantly obvious even to you that I'm far too busy in the hotel to be wasting time doing maths homework for the lodgers even if they are rolling in it. In any case it's been donkey's years since I did any proper maths. Besides it would be cheating if it's for a dissertation because he has to write that up on his own. Sorry dear but you'll just have to leave me out of your hare-brained schemes this time. (*turns back to continue typing more furiously*)

Sybil: Very well dear, if that's how you feel about it. He must be desperate though because he said he'd pay as much as five hundred pounds to anybody who can come up with the correct solution.

Basil: (stops typing suddenly, finger in mid air, sits up all ears and gulps). Did you say five hundred smackers? Well perhaps that... could put a slightly different complexion on it. For that sort of money I'd even have a go at proving Fermat's last theorem.

Sybil: It's already been done dear. Don't you read the newspapers?

Basil: Yes, but not the sort of trashy comic strips you buy dear, I find the written word far more effective for conveying ideas than all those pictures of false eyelashes and jars of face cream that you find so riveting.

Sybil: Very amusing I'm sure Basil. Anyway here are all the details in case you change your mind later on (*hands him a sheet of paper*). It shouldn't take a man of your prize-winning qualifications much more than half an hour or so. I told them you'd probably have the answer later this evening. I'm off to the hairdresser's now dear. See you later. (*exit Sybil*).

Basil looks down apprehensively at the piece of paper in his hand which reads:

"Dear Professor Fawlty. I am writing some computer-game software and would appreciate your help with a mathematical difficulty connected with avatar morphing. My problem is as follows:

Given the co-ordinates of N data points in a plane I need to find the equation of the quartic polynomial that exactly passes through the first and the last data points and best fits the remaining $N-2$ points in the least-squares sense (minimising the vertical deviations). I've enclosed some data points for a typical case with $N = 10$ (Figure 1).

The standard method for curve-fitting in all the text books is no use to me as far as I can tell because the best-fit curves are not guaranteed to pass exactly through the end points and in general will not do so. I need a formula for a curve that is guaranteed to always pass through the end points.

Thank you. (Room 7) "

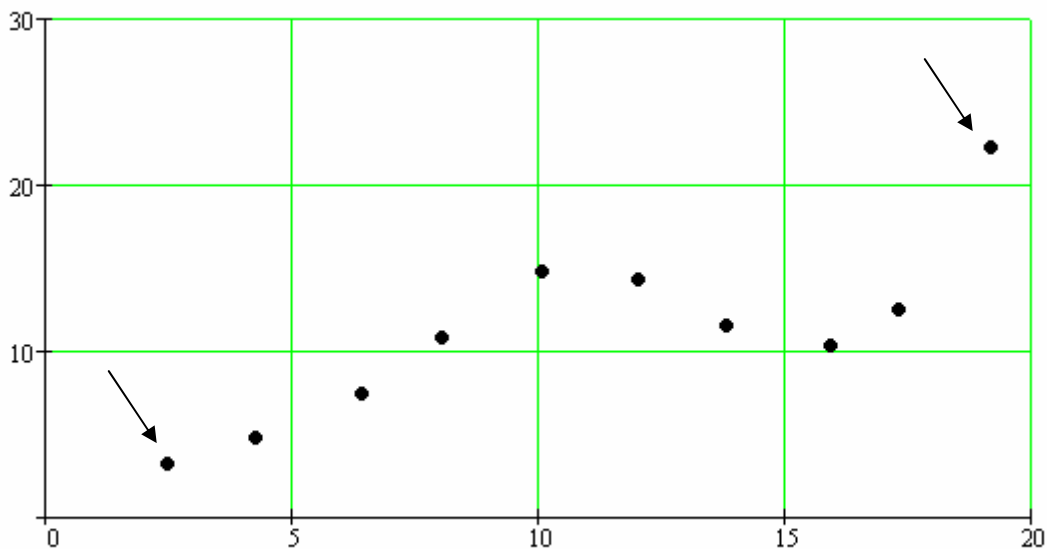


Figure 1

The fitted curve is required to pass through the two end points marked by the arrows.

(After reading the note Basil evidently engages a panic mode, leans over the typewriter, puts his head in his hands and groans out loud slowly rocking back and forth.)

Basil: (muttering to himself) Oh dear, Oh dear! If Sybil even so much as suspects I lied about going to University I'll probably hear the last of it sometime *after* pigs have learned to fly. I haven't the faintest idea what all this stuff means. What on earth is this 'Least Squares' business anyway? It wasn't on any of the 'O' level papers that I failed that's for sure². Oh....Oh My....*(head in hands – rocks back and forth..)*

Enter The Major (ex-army - retired), carrying a book of crossword puzzles³

Major: Morning Fawlty. Goodness me look at the time! Bar open yet is it? I could just manage a quick one before lunch. I say what are you looking so down in the dumps about Fawlty? One of the guests run off with the bar takings have they?

Basil: No, no it's worse than that Major. The Dragon Lady has gone and volunteered me to solve a maths problem for one of the guests and my wretched migraine has come on again so I just can't concentrate properly. Under normal circumstances of course I'd be able to knock it off in a jiffy but they want the solution by this evening and I doubt if I'll be right before then. It's this wretched shrapnel in my leg you know, sometimes gives me gyp for days on end. I don't suppose you know of anybody that might help out do you? There's a hun... hum er.. fifty quid in it for anybody can come up with a solution.

Major: (gulps – looks interested) Fifty smackers eh? Well as a matter of fact now you come to mention it I used to be a bit of a whiz at that sort of thing myself don't you know. Why during the war I helped to crack the Enigma code and all that but I mustn't talk about it because it's still very hush-hush even now. Give it here Fawlty, let's have a dekko. *(takes the sheet of paper from Basil and reads it carefully).*

Major (studies the paper): Why, this doesn't look too difficult at all old man. Quite an interesting problem in fact. Give me a bit of time in the bar and I'll knock it off for you no trouble. Get the cash ready old boy. I may try using Lagrange multipliers to find the condition for a constrained quartic least squares fit with the two end points anchored. I've not seen it done before but it shouldn't be all that difficult. Should be a bit of fun anyway and I've got my laptop here to help. It will pass the time nicely and be a sight easier than some of these Araucaria Jigsaw crosswords I'll be bound. *(The Major exits towards the bar and Basil perks up, visibly regaining his composure)*

Act Two.

Scene: The same lobby later in the day. Basil feeling pleased with himself for his anticipated financial coup stands impatiently behind the counter waiting for the Major, who enters from the bar bearing several sheets of paper.

Basil (eagerly) Well! well have you done it Major!?

Major: What? Oh yes! Of course old boy. Nothing to it really. Very interesting little puzzle it was too, thanks. I've given you extra value for money by generalizing the problem to fit polynomials of arbitrary degree p instead of just quartics. Of course you can just put $p = 4$ into the solution if you want to stick with the original quartic. Here, take a gander, you see I've minimized the sum of the squared vertical deviations but in the end I decided not to use Lagrange multipliers at all. Instead I built the end-point constraints directly in to the trial polynomial right at the start by changing the origin to coincide with the first end point and then choosing the form of the coefficients of the trial polynomial to force them always to also exactly pass through the last data point at the far end. I then differentiated in the usual way to minimize the sum of the squares of the vertical deviations. The interesting part was to try and write out the resulting set of linear equations in a neat concise way that will be valid for all values of p and N . Here it is. Just solve those p linear equations to obtain the coefficients of the best fit polynomial. I tried it out with $p = 4$ to find the best quartic. Battery went flat on the lap-top before I got chance to actually plot the fitted curve but it's guaranteed to pass through the end points nicely as required. There's no extra charge for the generalization, fifty will do nicely.

The Major's Solution: The original data points are $(x_i, y_i) \quad i = 1, 2..N$ It is assumed the x values are in ascending order so that $x_1 \leq x_2 \leq \dots \leq x_N$.

Shift the origin to the first data point

$$X_i = x_i - x_1$$

$$Y_i = y_i - y_1$$

Relative to the new co-ordinate system define the trial function $Y_{fit}(X)$ by

$$Y_{fit}(X) = X \left\{ \alpha_1 + \sum_{r=2}^p \alpha_r (X^{r-1} - X_N^{r-1}) \right\}$$

$Y_{fit}(X)$ is the trial polynomial of degree p constrained to satisfy the given condition to always pass through the new origin. The other given condition, that it must also pass

through the final (i.e. rightmost) data point (X_N, Y_N) , is met simply by choosing the value of α_1 so that $\alpha_1 = \frac{Y_N}{X_N}$, $X_N \neq 0$.

The value of the coefficient α_1 is thus known and to complete our task we need only to find the values of the remaining $p - 1$ unknown coefficients α_r , $r = 2, 3, \dots, p-1$ to minimize the sum of the squares of the vertical residuals, $S = \sum_{i=1}^N (Y_{fit}(X_i) - Y_i)^2$.

It's not too difficult to show (they always say that don't they? - See Appendix) that the vector α containing all the required coefficients is given by

$$\underline{\alpha} = \underline{M}^{-1} \underline{v}$$

where \underline{M} is the $p \times p$ matrix having elements given by

$$M_{q,r} = \sum_{i=1}^N X_i^1 (X_i^{r-1} - X_N^{r-1})(X_i^{q-1} - X_N^{q-1}) \quad (\text{Beware!})$$

$$M_{1,r} = \delta(1,r) = M_{r,1}$$

and the $p \times 1$ column vector \underline{v} has elements

$$v_1 = \frac{Y_N}{X_N}$$

$$v_q = \sum_{i=1}^N \frac{X_i}{X_N} (Y_i X_N - X_i Y_N) (X_i^{q-1} - X_N^{q-1})$$

$r = 1, 2, \dots, p$ and $q = 2, 3, \dots, p$

Basil: (admires the solution while smilingly handing over a fifty pound note from his wallet) Oh! Terrific Major! Well done Major, Very well done indeed, very impressive too with all those fancy looking squiggles as well. Doesn't seem a lot to it though does there for fifty quid, just a few lines? Couldn't you have stretched it out a bit? Made it look a bit more difficult? I'd have done it myself of course if it hadn't been for the old migraine but you know what these women are like. They don't understand all the agony we old campaigners have to put up with eh? They might even have said it was too hard for me. Ha! Ha! Not a word to Sybil mind. Mum's the word!

Excitedly Basil licks his pencil and begins copying out the solution in his own handwriting. The Major returns to the bar and some time later Sybil enters carrying some heavy shopping bags.

Sybil: Polly, would you be a dear and take these bags upstairs for me please, just a few dresses I picked up in town. Well, have you done it Basil?

Basil: Done what dear? The ironing? The accounts? All those jobs you should have done yesterday? Oh! you must mean that trifling little maths puzzle? Yes dear, of course I have. Dashed it off between making the beds and ordering the groceries. Here it is dear. I trust the ready cash will be following in due course. Fifties will do nicely thank you and then I'll be off out to celebrate if it's all the same to you thank you so very much. (*smirks as he hands over the hand-written sheets to Sybil*)

Sybil: (*glances at the solution and looks suitably impressed*) Very clever dear I must say! You're not just a pretty face after all are you? I must admit I had my doubts but you've surprised me this time. I'll go upstairs to Number 7 right now and get the money for you. (*exits*)

Act Three.

Scene: The Breakfast Room next morning. Manoel is serving the nice young couple from Number 7. The Major is eating alone at a nearby table and reading the newspaper. Sybil enters the room and begins to move towards the breakfast table of the Number 7 couple but Basil beats her to it, semi-discretely, elbowing her aside and unctuously rubbing his hands, he hovers over the nice young couple.

Basil: Good morning Mrs. Doctor, and Good Morning to you Sir. Are you enjoying your meal? Can I get you anything more? Anything at all, just say the word. More sausages? Eggs? Some more tea perhaps?

Number 7 man: (*angrily*) No Thank You! One load of tripe is enough from you for one day Mr. Fawlty thank you all the same.

Basil: (*looking nonplussed and indignant*) I beg your pardon! *Tripe?* Did you say *Tripe?* I'm sorry but I don't quite follow. This is a good English Breakfast and I can assure you there's not a drop of *Tripe* in it.

Number 7 man: I wasn't referring to the breakfast Mr. Fawlty. I was referring to that so-called *solution* your wife sold us yesterday. Talk about a pig in a poke. So you're the Manager are you? Mr. Basil Fawlty? When I spoke to you yesterday morning I hadn't realized you were the Manager, I thought you were the receptionist. I was looking for the Manager all last night to complain but they said he had gone out somewhere. Look here, I've plotted that so-called least-squares fit you sold me expecting it to lie at least somewhere close to the original data points and look at the load of rubbish that came out! (see Figure 2) I'd have done better to have asked the cat. The only thing your fitted curve does correctly is to pass through the end points. Apart from that it's a million miles away from the other data points! Part of it is invisible because it drops off the page at one point. I suspected something was wrong when I saw how short the solution was. Load of rubbish! *Tripe* in fact, as I just said! I must say expected a lot more than that for my thousand quid and I'd like my money back tout-de-suite if you don't mind.

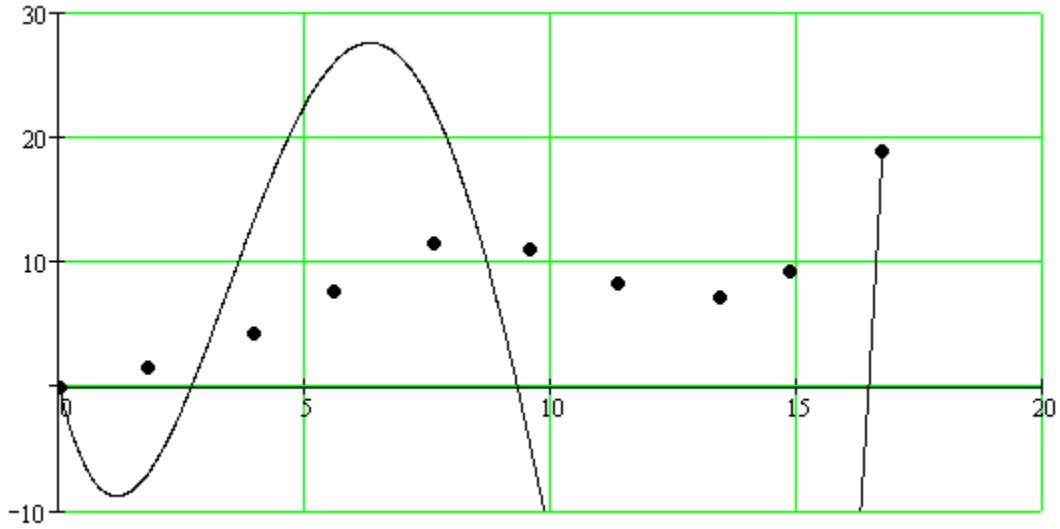


Figure 2
The Faulty Power makes a Dog's Breakfast

Basil: THOUS.....! (aghast and taken aback switches furious glances torn between the paper in his hand and Sybil who is beating a hasty retreat to the exit, wrings his hands and grits his teeth). I'm sure it's all just a simple little misunderstanding Sir. I'll have it all sorted out IMMEDIATELY as soon as I've had a word with my WIFE. Don't you worry Sir, enjoy your meal. SYBIL...!

Major: (approaches the table, stands in front of Basil to stop him from leaving and in the earshot of all says) Ah there you are Fawltly! Hang on a second old boy. It's about that little maths problem yesterday. I've charged my batteries and had another look at my solution this morning. Would you believe it but I've found an error in one of the powers. It's my laptop keyboard you know. It's not been the same since I spilled that gin on it last week and since then sometimes when the '2' key is pressed it goes and puts a '1' on the screen instead. Can't think why. Sorry about that, it's a dashed nuisance, but fortunately it's easy enough to put matters right; In the expression for the matrix element $M_{q,r}$ the bit that says X_i^1 just change it to read X_i^2 , otherwise you'll get a real dogs breakfast when you plot the graph. Here's the correct expression

$$M_{q,r} = \sum_{i=1}^N X_i^2 (X_i^{r-1} - X_N^{r-1}) (X_i^{q-1} - X_N^{q-1})$$

Who'd have thought it eh? A little mistake like that making such a big difference. Sorry about it old boy, but for fifty quid it's only fair you should have the correct answer. Still, no harm done I trust. See you later. Bye.

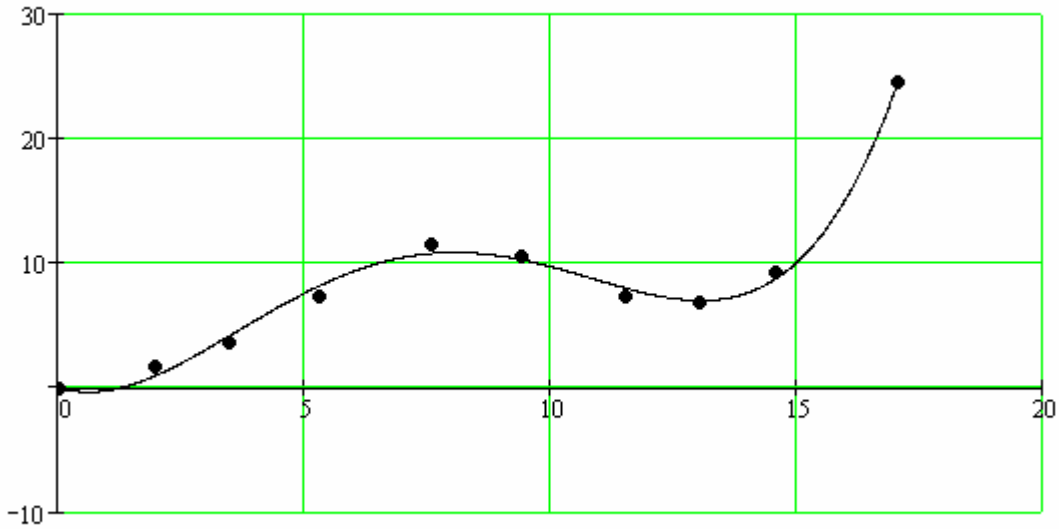


Figure 3

The points and the fitted quartic after the Faulty Power has been corrected⁴.

A furious and speechless Basil with arms outstretched in the manner of intended strangulation turns and rushes out after a retreating Sybil to discuss matters further while the Number 7 man stands up and hurries over towards the Major - intent on making him an offer he can't refuse. Manoel, still holding the teapot, looks on, thoroughly bemused by the whole proceedings. Screeching noises can be heard off. Closing Theme Tune.

Appendix

The derivation is quite straightforward although possibly it was not so obvious to Basil. Begin by writing out in full the expression for the quantity to be minimized:

$$S = \sum_{i=1}^N (Y_{fit}(X_i) - Y_i)^2 = \sum_{i=1}^N \left(X_i \left\{ \alpha_1 + \sum_{r=2}^p \alpha_r (X_i^{r-1} - X_N^{r-1}) \right\} - Y_i \right)^2$$

Now set to zero the derivatives of S with respect to each of the unknown coefficients α_q , where $q = 2, 3, \dots, p$. This gives

$$\frac{\partial S}{\partial \alpha_q} = 2 \sum_{i=1}^N \left[\left(X_i \left\{ \alpha_1 + \sum_{r=2}^p \alpha_r (X_i^{r-1} - X_N^{r-1}) \right\} - Y_i \right) \left[X_i^{q-1} - X_N^{q-1} \right] X_i \right] = 0$$

After rearranging and canceling the factor of 2 we have

$$\sum_{i=1}^N \sum_{r=2}^p \alpha_r X_i^2 (X_i^{r-1} - X_N^{r-1}) (X_i^{q-1} - X_N^{q-1}) = \sum_{i=1}^N X_i (Y_i - \alpha_1 X_i) (X_i^{q-1} - X_N^{q-1})$$

Substituting $\alpha_1 = \frac{Y_N}{X_N}$ it can be seen that we have obtained the required set of linear equations that are expressed in matrix form in the text.

0. Any resemblance to the fictional characters in a TV series of a similar sounding name is a purely accidental and if you believe this please contact me as soon as possible to discuss highly lucrative joint ventures.
1. Fawlty Towers, Second Series "Basil the Rat" starring John Cleese and Prunella Scales, first screened on BBC2, 25th October 1979.
2. Basil's 1963 'O' level papers may be found on <http://www.mathapps.com>
3. Book of Araucaria Crosswords, Paperback, 192 pages, 60 new Jigsaw Crosswords, Chambers Harrap Publishers Ltd, ISBN 0550101101.
4. The original data points (x_i, y_i) were generated on a whim by $y = xe^{\left(\frac{-x \sin(\frac{x}{2})}{30}\right)} + 1$.
The Mathcad document used to obtain Figures 2 and 3 is also available on <http://www.mathapps.com>. Fifty Pound notes should be posted to me in a plain envelope c/o PO Box.... (address deleted Ed.) and I will hand them over to the Major for ~~my~~ his retirement fund.