UNIVERSITIES OF MANCHESTER LIVERPOOL
LEEDS SHEFFIELD AND BIRMINGHAM

Joint Matriculation Board

General Certificate of Education

FURTHER MATHEMATICS

SPECIAL PAPER

FRIDAY 21 June 1963, 9.30-12.30

Negligently presented or slovenly work will be penalized. *Answer* **six** *questions.*

1. (*a*) The complex numbers z_1, z_2, z_3 are represented by the points *A, B,* C respectively in the Argand diagram. If

$$2z_1^2 + z_2^2 + z_3^2 - 2z_3z_1 - 2z_1z_2 = 0$$

show that *AB* and *AC* are equal and perpendicular to each other.

(*b*) State and prove de Moivre's theorem in the case when the index *n* is a positive integer.

Show that the roots of the equation

$$x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = 0$$

are $\cos\frac{r\pi}{9} + i\sin\frac{r\pi}{9}$ where *r* = 1.3,5,7,11,13, 15, 17.

Deduce that

$$\cos\frac{\pi}{9} + \cos\frac{3\pi}{9} + \cos\frac{5\pi}{9} + \cos\frac{7\pi}{9} = \frac{1}{2}$$

2. Find the number of ways in which n identical coins can be placed (i) in two different boxes, (ii) in three different boxes, all the coins being used and empty boxes being allowed in each case. Show that the number of ways in which the coins can be placed in four different boxes is

$$\frac{1}{2}\sum_{s=0}^{n}(s+1)(s+2)$$

1f ${}_nF_r$ denotes the number of ways in which the n coins can be placed in r different boxes, show, by considering the cases when one particular box is empty or has at least one coin, or otherwise, that

$${}_nF_r = {}_{n-1}F_r + {}_nF_{r+1}$$

Use this relation to construct a table of values of ${}_nF_r$ for $n = 1, 2, 3, 4, 5$ and r 1, 2, 3, 4, 5.

3. If s is the length of arc of a curve $y = f(x)$ measured from an arbitrary point on it, show that

$$(ds/dx)^2 = 1 + (dy/dx)^2$$

In a certain curve which passes through the origin the length of arc satisfies the relation

$$s = a\log_e\frac{a}{a-x}, \quad (a > 0).$$

Show that the curve must lie between the lines $x = 0$ and $x = a$ and that it can be represented parametrically by the equations

$$x = a(1-\cos t).$$
$$y = a\log_e(\sec t + \tan t) - a\sin t.$$

Sketch the complete curve in the range $-\frac{1}{2}\pi \le t \le \frac{1}{2}\pi$

If the curve makes a complete revolution about the line $x = a$, show that the area of the surface generated is $4\pi a^2$.

4. A line of gradient k passes through the point $A(0, a)$ and cuts the fixed line $y = c$, where $c > 0$, in the point P. Find the coordinates of P. The line through A of gradient $1/k$ cuts the line OP in the point Q, O being the origin. Show that for all values of k the point Q lies on the curve whose equation is

$$cx^2 + (a-c)y^2 - a(a-c)y = 0 .$$

Describe how this curve changes its character as the point A takes successive positions on the positive part of the y-axis, stating the name of the curve for all values of $a > 0$, and for $a = 0$. Show in particular that when a = $5c$ the curve is an ellipse with semi-axes $5c$, 5c/2.

5. A body A of mass M is connected by a light inextensible string to a vessel B containing liquid, the mass of the vessel and liquid being M at time $t = 0$. The string passes over a fixed smooth peg below which A and B hang and the system is initially at rest. The liquid evaporates in such a way that its mass decreases at a constant rate μ per unit time. Find the acceleration of the system at time t, and deduce that its speed is then

$$-gt - \frac{2Mg}{\mu}\log_e\frac{2M-\mu t}{2M} .$$

Obtain an approximation to this expression when $\mu\, t/M$ is small, neglecting powers of t higher than the second. Deduce that the distance through which A has fallen at time t is at^3 approximately. where a is a constant, and find the value of a in terms of M, μ . and g. If the mass which evaporates in one day is $\frac{1}{2}M$, find the approximate distance through which A has fallen at time t = 60 sec., taking the value of g to be *32* ft./sec^2.

6. A particle P of mass m moves in the x, y-plane under the action of forces **R** and **S** which are respectively tangential and normal to its path. These forces have magnitudes $mk\omega$ and $m\lambda\omega$ respectively, where ω is the speed of P, and k, A are constants. The tangential force **R** opposes the motion and **S** is directed so that rotation through a right angle from **R** to **S** is in the same sense as that from the x-axis to the y-axis. Show that if u, v are the x- and y-components of the velocity of P at time t, the equations of motion are

$$du/dt = \lambda v - ku\,, \quad dv/dt = -kv - \lambda u$$

Deduce that

$$\frac{d^2u}{dt^2} + 2k\frac{du}{dt} + (\lambda^2 + k^2)u = 0$$

If $u = 0$, $v = v_0$ when $t = 0$, show that at time t

$$u = v_o e^{-kt}\sin\lambda t\,, \qquad v = v_o e^{-kt}\cos\lambda t$$

7. Two particles move in a plane under the action only of the mutual force between them. Prove that the sum of the components of their momenta in any direction is constant.

Two atomic nuclei of masses $3m$ and m move along perpendicular lines AO, BO with speeds u. $2u$ respectively and collide at O. On collision they are transformed into two nuclei of masses m and $4m$ which move with speeds v_1, v_2 respectively along a line XOX', where OX lies in the right angle A$0B$. Show that tan AOX = 4/3. If at the collision the total kinetic energy is decreased by $\frac{1}{2}mu^2$, find the possible values of v_1, v_2.

Show also that in all cases in which the transformed nuclei move along XOX' after the collision, the loss of energy at the collision cannot exceed $3mu^2$.

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8. The angular momentum h about the origin of a particle P of mass m moving in a plane is given by

$$h = m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

where x, y are the coordinates of P at time t. Deduce that the moment about the origin of the resultant force acting on P is equal to dh/dt.

A uniform rod of mass M and length $2a$ is freely pivoted at its centre to a fixed point, and is initially at rest in a horizontal position. A particle of mass m. released from rest at a height c above the rod, strikes it at a distance x from the pivot and adheres to it. Find the angular velocity of the rod immediately after the impact. Show that the rod will subsequently make complete revolutions if

$$c > \frac{3mx^2 + Ma^2}{3mx} \qquad (1)$$

Show that this condition cannot de satisfied for any value of x unless $c > 2a\sqrt{(M/3m)}$.

If $M > 3m$, show that the condition (1) cannot be satisfied unless $c > (3m + M)a/3m$.

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