UNIVERSITY OF NOTTINGHAM

FACULTY OF PURE SCIENCE

FIRST YEAR PART I EXAMINATION, 1966

ORDINARY DEGREE EXAMINATION (A), 1966

MATHEMATICS I (ia)

FRIDAY *June* 10*th* 9-45 -- 12-45

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1. Prove that if a is a complex root of a polynomial equation with real coefficients then so also is $\bar{a}$.

Show that $z^{2n}+1$ has real quadratic factors given by

$$z^{2n+1} = \prod_{k=1}^{n}\left(z^2 - 2z\cos\frac{\pi}{2n}(2k+1)+1\right).$$

Hence prove that

$$\cos(n\theta) = 2^{n-1}\prod_{k=1}^{n}\left(\cos\theta - \cos\frac{\pi}{2n}(2k+1)\right).$$

2. (*a*) Show that for any two complex numbers z_1 and z_2

$$|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$$

Hence show that the locus of points on the Argand diagram such that

$$|z|^2 + 2\,\text{Re}(iz) = 3$$

is a circle, and give its centre and radius. Show also that the locus cuts the circle $|z|^2 = 3$ on the real axis and find the points of intersection.

(*b*) Sum the series

$$\sum_{n=1}^{\infty}\frac{\cos n\theta}{n!} \quad (\theta \text{ real})$$

3. (*a*) Express the following in the form *a* +*ib* where *a* and *b* are real, stating principal values where defined:

(i) $\sinh(3+i)$ (ii) $\log(3-4i)$ (iii) $(\sqrt{3}+i)^i$

(b) Show that

$$\tan^{-1} z = k\pi + \frac{1}{2}\theta + \frac{1}{4} i \log \frac{x^2+(1+y)^2}{x^2+(1-y)^2} \quad z \neq \pm\, i$$

where $z = x+iy$, *k* is any integer, and where θ is such that

$$\cos\theta : \sin\theta : 1 : \; : 1-x^2-y^2 : 2x : \sqrt{\left(1-x^2-y^2\right)^2+4x^2} \, .$$

4. (*a*) $|a_n| < K|b_n|$ for all large *n*, where *K* is a positive constant. Prove that if $\sum_1^\infty |b_n|$ converges then so does $\sum_1^\infty |a_n|$. Does the series $\sum_1^\infty a_n$ also converge?

(*b*) Investigate the convergence (including absolute convergence) of the following series:

(i) $\sum_{n=1}^{\infty} \frac{n}{4n^3-2}$ (ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^2+1}$

(iii) $\sum_{n=1}^{\infty} \frac{n(x-1)^n}{2^n(3n-1)}$ (*x* real) (iv) $\sum_{n=1}^{\infty} \sin^3 \frac{1}{n}$

(5). State Taylor's theorem with Lagrange's form of the remainder.

Hence, by writing log(1+*x*) as a sum of three terms, prove that if *n* is a positive integer greater than 2 then

$$0 < \frac{2}{n} - \log\left(1+\frac{2}{n}\right) < \frac{2}{n^2}$$

Hence, or otherwise, show that

$$\frac{1}{2n+1} + \frac{1}{2n+3} + \frac{1}{2n+5} + \ldots + \frac{1}{4n+1} \to \frac{1}{2}\log 2$$

6. (*a*) Compute the following limits:

(i) $\lim_{x\to 0}\frac{x-\sin x}{x^3}$ (ii) $\lim_{x\to 0}\frac{e^{2x}-2e^x+1}{\cos 3x-2\cos 2x+\cos x}$

(iii) $\lim_{x\to 0}\frac{3^x-2^x}{x}$

(*b*) If

$$f(x)=(\sin^{-1}x)^2$$

prove that

$$(1-x^2)f''(x)-xf'(x)=2$$

and that

$$(1-x^2)f^{n+1}(x)-(2n-1)xf^n(x)-(n-1)^2f^{n-1}(x)=0$$

Assuming that $(\sin^{-1}x)^2$ may be expanded as an infinite power series about the origin show that

$$(\sin^{-1}x)^2=\sum_{n=1}^{\infty}\frac{2^{2n-1}\{(n-1)!\}^2}{(2n)!}x^{2n}$$

7. (*a*) Prove that if $f(x_1,x_2,....,x_n)$ is a homogeneous function of degree *m*, then

$$\sum_{r=1}^{n}x_r\frac{\partial f}{\partial x_r}=mf\ ,$$

provided the derivatives exist.

If $V=V(x,y,z)$ be a homogeneous function of degree *m* having continuous second partial derivatives, prove that

$$x^2V_{xx}+y^2V_{yy}+z^2V_{zz}+2xyV_{xy}+2yzV_{yz}+2zxV_{zx}=m(m-1)V$$

(*b*) If $u=x^3y$ and *x* and *y* are defined implicitly as functions of *t* by the relations

$$x^5+y=t\ ,\qquad x^2+3y^3=t^2\ ,$$

show that

$$\frac{du}{dt}=\frac{33}{43}\ \text{when } x=1 \text{ and } y=1.$$

[*Turn over*

8. Show that a solution of the partial differential equation

$$3\frac{\partial^2 u}{dx^2} = \frac{\partial u}{\partial t}$$

for $t > 0$ and $0 < x < 2$, subject to the boundary conditions

$$u(0,t) = u(2,t) = 0\,; \qquad u(x,0) = x$$

is given by

$$u(x,t) = 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \exp\left(-\frac{3}{4}n^2\pi^2 t\right) \sin\frac{1}{2}n\pi\, x$$

9. Find two linearly independent power series solutions of the equation

$$x^2\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + (2 + x^2)y = 0$$

which are valid for small x, giving the general term in each case.

[455]