

UNIVERSITY OF NOTTINGHAM

FACULTY OF PURE SCIENCE

FIRST YEAR PART I EXAMINATION (A), 1966

ORDINARY DEGREE EXAMINATION (A), 1966

MATHEMATICS I (iia)

SATURDAY *June 11th* 9-45 - 12.45

1. Show that the system

$$\begin{aligned}X - Y + 2Z - \lambda T &= 4 + \lambda \\3X + \lambda Y + 2Z &= 6 \\3X + 3Y - 2Z - +2\lambda T &= -2\lambda \\2X + Y + \lambda Z + 3\lambda T &= 2 + 4\lambda\end{aligned}$$

is inconsistent for one and only one value of λ , that it has a non-unique solution for exactly one other value of λ , and in all other cases, has a unique solution.

Find the complete solution in the non-unique case, indicating the rank and nullity of the corresponding linear map and find the unique solution for which $T = 1$.

2. Test the following sets of vectors for linear dependence, and find a basis for the subspace spanned by these:

(i) $(0, -1, 0, 3, 6, 2)$, $(2, 4, 3, -1, -2, 2)$, $(1, 1, 2, 1, 3, 1)$;

(ii) $(2, 1, 3, -1)$, $(-1, 1, -2, 2)$, $(1, 5, 0, 4)$.

Extend the latter set to form a basis for V_4 .

[*Turn over*

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3. Let \mathbf{A} , \mathbf{B} be the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^4 & b^4 & c^4 \end{bmatrix}.$$

Prove that

$$\det \mathbf{A} \det \mathbf{B} = (a-b)(b-c)(c-a)$$

and that

$$\det \mathbf{A} \det \mathbf{B} = \begin{bmatrix} s_0 & s_1 & s_2 \\ s_2 & s_3 & s_4 \\ s_4 & s_5 & s_6 \end{bmatrix}$$

where

$$s_r = a^r + b^r + c^r \quad (r = 0, 1, \dots, 6)$$

Factorise the last determinant as a product of linear factors.

4. Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find a matrix \mathbf{P} such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

5. Solve the differential equations

$$(a) \frac{dy}{dx} = \frac{y-x}{8y-x+7}$$

$$(b) \frac{dy}{dx} \cos x + y \sin x = 2x \sin 2x + 2x^2.$$

6. Show that the substitution $y = zx^{-1/2}$ transforms the differential equation

$$4x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + (4x^2 - 1)y = 4x^{3/2} e^x \sin x$$

into one with constant coefficients, and hence obtain the general solution of the given equation.

7. Solve the equations

$$(a) \quad \frac{dy}{dx} + y = y^2 (\cos x - \sin x)$$

$$(b) \quad x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = x + x^2 \log|x| + x^3$$

8. State the Dirichlet conditions under which the Fourier series corresponding to a function $f(x)$ defined in $(a, a+2T)$ converges.

Expand the function $f(x) = x^2$ in a Fourier series in the range $0 < x < 2\pi$, and draw a diagram indicating the sum of the resulting series for all values of $x \in [-4\pi, 4\pi]$.

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

9. Assuming that the Legendre polynomial $P_n(x)$ is defined by the expansion

$$(1 - 2xh + h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)h^n \quad (|x| \leq 1, |h| < 1),$$

prove that

$$(i) \quad (n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0,$$

and hence evaluate

$$\int_{-1}^1 xP_n(x)P_{n+1}(x)dx;$$

$$(ii) \quad xP_n'(x) - P_{n-1}'(x) = nP_n(x)$$

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