The Great Weighted Wheel

Introduction.

The outline analysis given below is intended to be read in conjunction with the article "The Great Weighted Wheel" in Mathematics Today, where the full analysis is given. The notation and definitions used in this mathcad document are the same as in that paper.

The great weighted wheel was constructed in the Tower of London circa 1640 under the direction of Edward Somerset, the Second Marquis of Worcester. The intention was to obtain perpetual motion.

In this mathcad document an expression for the torque and potential energy of the wheel in angular position α is obtained. It is shown that there is a stable equilibrium position for the wheel so that peprpetual motion will not be achieved. As a check the calculation is done twice by two slightly different procedures to confirm that the answers agree

Graphs of torque and potential energy are given as a function of the wheel's rotation. The user can change a parameter 'n' at the places indicated to locate the equilibrium position of the wheel.

The first method: This method is the same as the one decribed in the Mathematics Today article.

The angular displacement, α, of the wheel is measured with respect to a starting reference position,$\alpha = 0$, at which the spoke numbered zero is placed at the angle $\beta = \beta_1$ (see below).

Radius of outer wheel, R $R := 7$

Radius of inner wheel, r $r := 6$

length of string, a $a := 1$

number of masses, N $N := 40$

angular separation of spokes $\theta := \frac{2 \cdot \pi}{N}$

d_{1sq} and d_{2sq} are the squares of two radii used in the analysis:

$$d_{1sq} := a^2 + r \cdot R \cdot \cos(\theta) + r \cdot R \cdot \sin(\theta) \cdot \sqrt{\frac{4 \cdot a^2}{\left(r^2 + R^2 - 2 \cdot r \cdot R \cdot \cos(\theta)\right)} - 1}$$

$$d_{2sq} := \left(a^2 + r \cdot R \cdot \cos(\theta)\right) - r \cdot R \cdot \sin(\theta) \cdot \sqrt{\frac{4 \cdot a^2}{\left(r^2 + R^2 - 2 \cdot r \cdot R \cdot \cos(\theta)\right)} - 1}$$

$d_1 := \sqrt{d_{1sq}}$ $d_2 := \sqrt{d_{2sq}}$

$d_1 = 6.996$ $d_2 = 6.002$

Define some angles used in the analysis:

$$\lambda := \operatorname{acos}\left[\frac{\left(R^2 + d_1^{\,2} - a^2\right)}{2 \cdot R \cdot d_1}\right]$$ $\lambda = 8.194\,\text{deg}$

$$\mu := \operatorname{acos}\left[\frac{\left(R^2 + d_2{}^2 - a^2\right)}{2 \cdot R \cdot d_2}\right] \qquad \mu = 0.559\,\text{deg}$$

$$\varepsilon := \operatorname{acos}\left[\frac{\left(a^2 + d_2{}^2 - r^2\right)}{2 \cdot a \cdot d_2}\right] \qquad \varepsilon = 85.106\,\text{deg}$$

$$\chi := \operatorname{acos}\left[\frac{\left(a^2 + R^2 - d_2{}^2\right)}{2 \cdot a \cdot R}\right] \qquad \chi = 3.356\,\text{deg}$$

$$\eta := \operatorname{acos}\left[\frac{\left(a^2 + d_1{}^2 - r^2\right)}{2 \cdot a \cdot d_1}\right] \qquad \eta = 4.839\,\text{deg}$$

$$\omega := \operatorname{acos}\left[\frac{\left(r^2 + d_1{}^2 - a^2\right)}{2 \cdot r \cdot d_1}\right] \qquad \omega = 0.806\,\text{deg}$$

$$\gamma := \operatorname{acos}\left[\frac{\left(a^2 + R^2 - d_1{}^2\right)}{2 \cdot a \cdot R}\right] \qquad \gamma = 85.665\,\text{deg}$$

$$\phi := \frac{\pi}{2} - \gamma \qquad \phi = 4.335\,\text{deg}$$

define the 4 critical β values:

$$\beta_4 := \phi \qquad \beta_4 = 4.335\,\text{deg}$$

$$\beta_1 := \frac{\pi}{2} + \chi \qquad \beta_1 = 93.356\,\text{deg}$$

$$\beta_2 := \pi + \phi \qquad \beta_2 = 184.335\,\text{deg}$$

$$\beta_3 := 3\cdot\frac{\pi}{2} + \chi \qquad \beta_3 = 273.356\,\text{deg}$$

$$\beta_0 := \beta_1 \qquad j := \sqrt{-1}$$

Define the function Z(α) in two parts, Zlow and Zhigh:

$$\text{Zlowimag}(\alpha) := -18\cdot j\cdot a + j\cdot\csc\left(\frac{\theta}{2}\right)\cdot\left(a\cdot\sin\left(\alpha - \frac{\theta}{2}\right) + d_2\cdot\cos\left(\alpha + \chi + \mu + \frac{\theta}{2}\right.\right.$$

$$\text{Zlowreal}(\alpha) := \csc\left(\frac{\theta}{2}\right)\cdot\left(a\cdot\cos\left(\alpha - \frac{\theta}{2}\right) - d_2\cdot\sin\left(\alpha + \chi + \mu + \frac{\theta}{2}\right) + r\cdot\sin\left(\alpha\right.\right.$$

$$\text{Zlow}(\alpha) := \text{Zlowreal}(\alpha) + \text{Zlowimag}(\alpha)$$

$$\text{Zhighimag}(\alpha) := -20\cdot j\cdot a + j\cdot\csc\left(\frac{\theta}{2}\right)\cdot\left(a\cdot\sin\left(\alpha - \frac{\theta}{2}\right) + d_2\cdot\cos\left(\alpha + \chi + \mu -\right.\right.$$

$$\text{Zhighreal}(\alpha) := \csc\left(\frac{\theta}{2}\right)\cdot\left(a\cdot\cos\left(\alpha - \frac{\theta}{2}\right) - d_2\cdot\sin\left(\alpha + \chi + \mu - \frac{\theta}{2}\right) + r\cdot\sin\left(\alpha\right.\right.$$

$$Zhigh(\alpha) := Zhighreal(\alpha) + Zhighimag(\alpha)$$

and finally

$$Z(\alpha) := if[0 \le \alpha < (\phi - \chi), Zlow(\alpha), Zhigh(\alpha)]$$

Plot torque against α, torque = Re(Z(α))

m is the number of steps to be used over the 9 degree rotation

$$m := 180$$

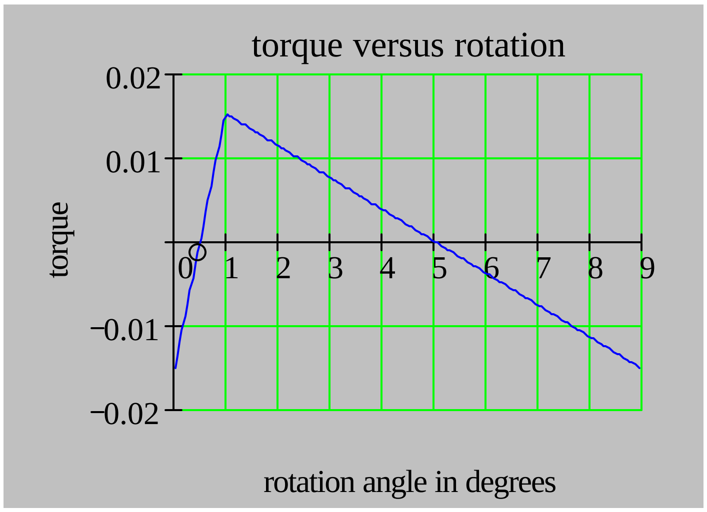
$$j := 0, 1 .. m - 1 \qquad h_j := \frac{\theta}{m} \cdot j$$

Store the m values of Z(α) in the vector called ans

$$ans_j := Z(h_j)$$

The black circle on the graph below marks the torque at n mths. of θ degrees. You can change the value of n to move the circle and find the values of n where the torque is zero. Increase the value of m (above) if higher resolution is required.

$$n := 9$$



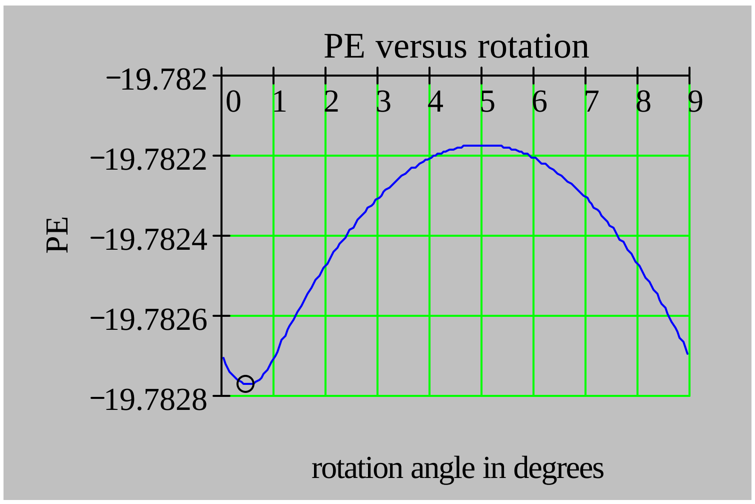
the black circle is at the angular position h_n

$$h_n = 0.45\,deg$$

Plot potential energy against a, PE = Im(Z(a))

$n := 9$

The black circle on the graph below marks the potential at n mths. of θ degrees. You can change the value of n to move the circle and find the value of n where the potential energy is a minimum.Increase the value of m if higher resolution is required.



the black circle is at the angular position h_n

$h_n = 0.45\,\text{deg}$

Maximum and Minimum values of Torque and PE

The potential energy is an extremum (dV/dα = 0) at the two values of α given by

$$\alpha_{min} := -\text{atan}\left(\frac{d_2 \cdot \sin\left(\chi + \mu + \frac{\theta}{2}\right) - r \cdot \sin\left(\chi - \frac{\theta}{2}\right) - a \cdot \cos\left(\frac{\theta}{2}\right)}{d_2 \cdot \cos\left(\chi + \mu + \frac{\theta}{2}\right) - r \cdot \cos\left(\chi - \frac{\theta}{2}\right) - a \cdot \sin\left(\frac{\theta}{2}\right)}\right)$$

$\alpha_{min} = 0.49\,\text{deg}$

$$\alpha_{max} := -\operatorname{atan}\left(\frac{d_2 \cdot \sin\left(\chi + \mu - \frac{\theta}{2}\right) - r \cdot \sin\left(\chi - \frac{3 \cdot \theta}{2}\right) - a \cdot \cos\left(\frac{\theta}{2}\right)}{d_2 \cdot \cos\left(\chi + \mu - \frac{\theta}{2}\right) - r \cdot \cos\left(\chi - \frac{3 \cdot \theta}{2}\right) - a \cdot \sin\left(\frac{\theta}{2}\right)}\right)$$

$\alpha_{max} = 4.99\,\text{deg}$

Plot torque against potential energy as the wheel is rotated anticlockwise through θ = 9^{o}:

To do this, plot the locus of Z(α) in the complex plane as α moves through 9^{o}

$j := 0, 1 .. \text{rows}(ans) - 1$

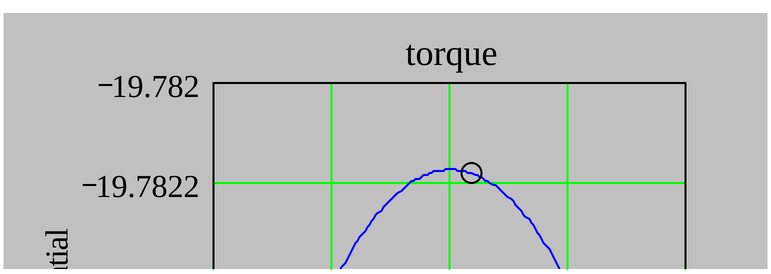
$\text{potential}_j := \text{Im}(\text{ans}_j)$ $\qquad$ $\text{torque}_j := \text{Re}(\text{ans}_j)$

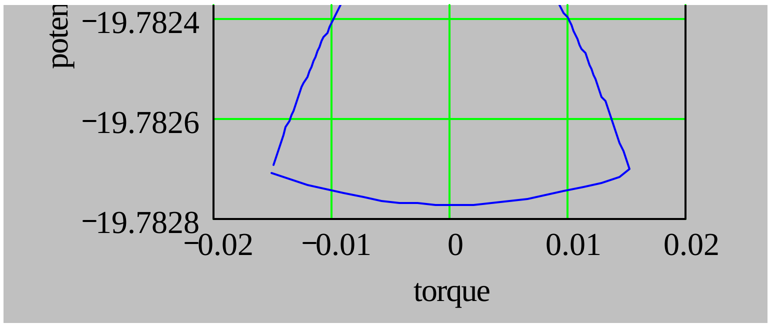
increase the value of n (0<n<179) to rotate anticlockwise from the start position in 1/20 degree steps

the value n = x corresponds to x/20 degrees anticlockwise:

change n below:

$n := 90$ $\qquad$ $\text{rotation} := \frac{n}{m} \cdot \frac{\theta}{\pi} \cdot 180$ $\quad$ $\text{rotation} = 4.5$ $\quad$ deg





Calculation of Z(α) by direct summation of the N terms (used as a check)

$$sum1(n1,n2,\alpha) := \sum_{n=0}^{n1} d_2\cdot \exp\left[i\cdot\left(n\cdot\theta + \mu + \beta_1 + \alpha\right)\right]$$

$$sum2(n1,n2,\alpha) := \sum_{n=n1+1}^{19} \left[r\cdot \exp\left[i\cdot\left(n\cdot\theta - \theta + \beta_1 + \alpha\right)\right] - i\cdot a\right]$$

$$sum3(n1,n2,\alpha) := \sum_{n=20}^{n2} d_1\cdot \exp\left[i\cdot\left(n\cdot\theta - \lambda + \beta_1 + \alpha\right)\right]$$

$$sum4(n1,n2,\alpha) := \sum_{n=n2+1}^{39} \left[R\cdot \exp\left[i\cdot\left(n\cdot\theta + \beta_1 + \alpha\right)\right] - i\cdot a\right]$$

$$\text{sumtotal}(n1,n2,\alpha) := \text{sum1}(n1,n2,\alpha) + \text{sum2}(n1,n2,\alpha) + \text{sum3}(n1,n2,\alpha$$

$$\text{Znew}(n1,n2,\alpha) := \text{sumtotal}(n1,n2,\alpha)$$

Now check to see if the direct sum Znew(n1,n2,α) gives the same values as the simplified version Z(α)

$$\text{Ztest}(\alpha) := \text{if}[0 \le \alpha < (\phi - \chi), \text{Znew}(10,30,\alpha), \text{Znew}(9,29,\alpha)]$$

$\alpha_{test} := .1$ input a test angle to use, in radians between 0 and 0.157 (i.e 0 to 9 degrees)

$$Z(\alpha_{test}) = -2.813 \times 10^{-3} - 19.782i$$

These answers should agree

$$\text{Ztest}(\alpha_{test}) = -2.813 \times 10^{-3} - 19.782i$$

The second method:

What follows is a slightly different approach, to the problem using less algebraic simplification. It essentially obtains again the same results as obtained in the first method:

Radius of outer wheel, R	$R := 7$
Radius of inner wheel, r	$r := 6$
length of string, a	$a := 1$
number of masses, N	$N := 40$
angular separation of spokes	$\theta := \frac{2 \cdot \pi}{N}$

d1sq and d2sq are the squares of two radii used in the analysis:

$$d1sq := a^2 + r \cdot R \cdot \cos(\theta) + r \cdot R \cdot \sin(\theta) \cdot \sqrt{\frac{4 \cdot a^2 - \left(r^2 + R^2 - 2 \cdot r \cdot R \cdot \cos(\theta)\right)}{r^2 + R^2 - 2 \cdot r \cdot R \cdot \cos(\theta)}}$$

$$d2sq := a^2 + r \cdot R \cdot \cos(\theta) - r \cdot R \cdot \sin(\theta) \cdot \sqrt{\frac{4 \cdot a^2 - \left(r^2 + R^2 - 2 \cdot r \cdot R \cdot \cos(\theta)\right)}{r^2 + R^2 - 2 \cdot r \cdot R \cdot \cos(\theta)}}$$

$$d1 := \sqrt{d1sq} \qquad d2 := \sqrt{d2sq}$$

$$\lambda := \mathrm{acos}\left[\frac{\left(R^2 + d1^2 - a^2\right)}{2 \cdot R \cdot d1}\right] \qquad \lambda = 8.194\,\mathrm{deg}$$

$$\mu := \mathrm{acos}\left[\frac{\left(R^2 + d2^2 - a^2\right)}{2 \cdot R \cdot d2}\right] \qquad \mu = 0.559\,\mathrm{deg}$$

Define below some functions that are used:

$$\mathrm{invcos}(x, y, z) := \mathrm{acos}\left[\frac{\left(x^2 + y^2 - z^2\right)}{2 \cdot x \cdot y}\right]$$

$$\mathrm{invtan}(x, y, z) := \mathrm{atan}\left[\frac{(x \cdot \sin(y) - z)}{x \cdot \cos(y)}\right]$$

$$invsin(x,y,z) := asin\left[\frac{(x^2 + y^2 - z^2)}{2 \cdot x \cdot y}\right]$$

define the 4 critical β values:

$$\beta 4 := invsin(a, R, d1)$$ $\beta 4 = 4.335$

$$\beta 1 := \pi - invsin(a, R, d2)$$ $\beta 1 = 93.356$

$$\beta 2 := \frac{(3 \cdot \pi)}{2} - invcos(a, d2, r) - invcos(R, d2, a)$$ $\beta 2 = 184.33$

$$\beta 3 := \frac{(3 \cdot \pi)}{2} - invcos(a, d1, r) - invcos(r, d1, a) + \theta$$ $\beta 3 = 273.35$

$$\beta 0 := asin\left(\frac{a}{R}\right)$$ $$j := \sqrt{-1}$$ $$\beta 0 := \beta 1$$

$$Z3(\beta) := d1 \cdot exp[j \cdot (\beta - \lambda)]$$

$$Z2(\beta) := if[\beta 2 \le (mod(\beta, 2 \cdot \pi)) < \beta 3, r \cdot exp[j \cdot (\beta - \theta)] - j \cdot a, Z3(\beta$$

$$Z1(\beta) := if[\beta 1 \le (mod(\beta, 2 \cdot \pi)) < \beta 2, d2 \cdot exp[j \cdot (\beta + \mu)], Z2(\beta)]$$

$$Z(\beta) := if(\beta 4 \le mod(\beta, 2 \cdot \pi) < \beta 1, R \cdot exp(j \cdot \beta) - j \cdot a, Z1(\beta))$$

$$Ztotal(\alpha) := \sum_{n=0}^{N-1} Z(\beta 0 + n \cdot \theta + \alpha)$$

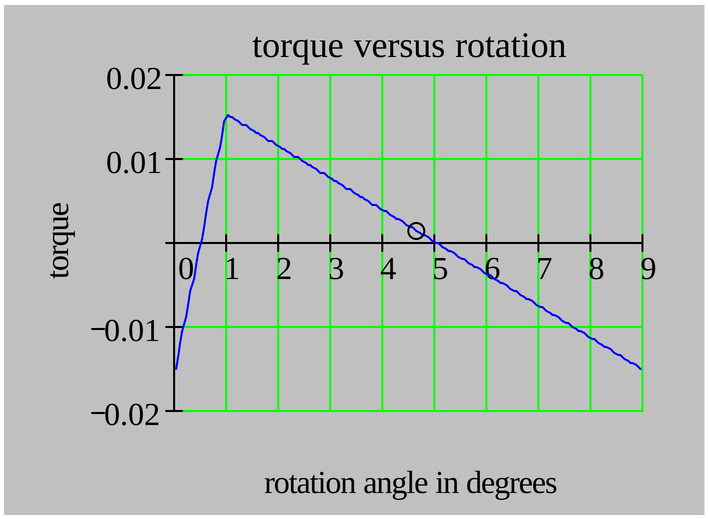
$$m := 180$$

$j := 0, 1 .. m - 1$ $h_j := \frac{\theta}{m} \cdot j$

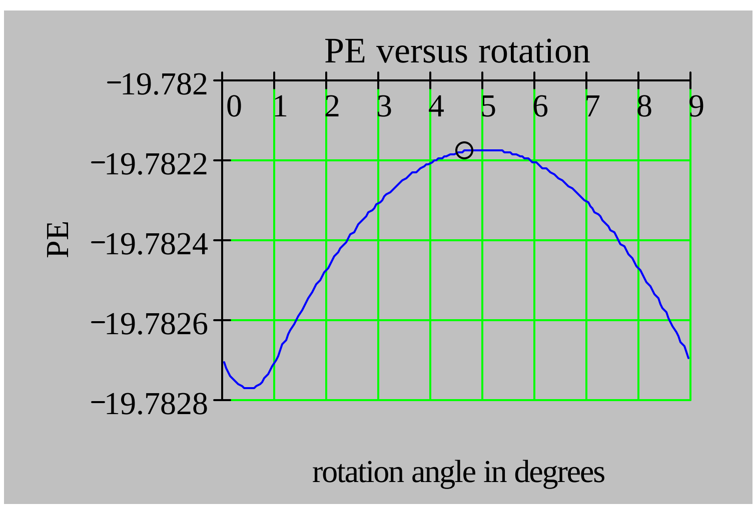
$ans_j := Ztotal(h_j)$

$r := 93$

The black circle below marks the torque at r m
degrees



$r := 93$



Plot the locus of the masses:

$j := 0, 1 .. 359$

$k := 0, 1 .. N - 1$

$x_j := 2 \cdot \frac{\pi}{359} \cdot j$

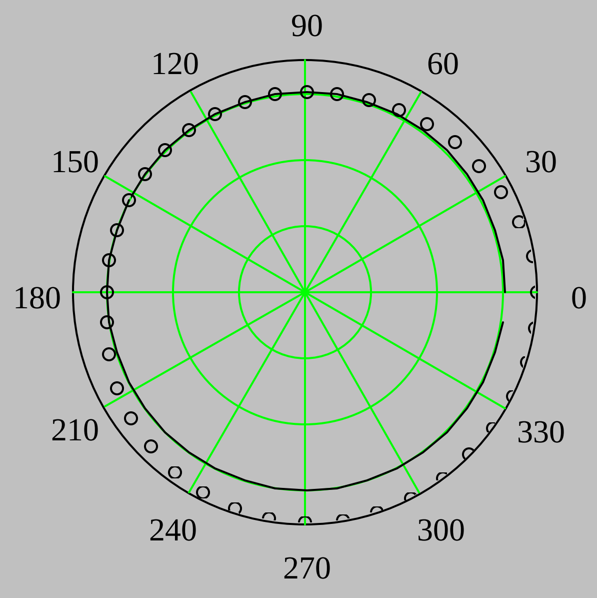
$y_k := 2 \cdot \frac{\pi}{N} \cdot k$

change β0 here: $\beta 0 := 0$ define β0 here

$X_k := \mathrm{Re}(Z(\beta 0 + y_k))$

$Y_k := \mathrm{Im}(Z(\beta 0 + y_k))$

$\mathrm{radius}_k := \sqrt{\left[(X_k)^2 + (Y_k)^2 \right]}$



locus of masses

The above position is the starting position for the anticlockwise rotation through α, $0<\alpha<\theta = 9^o$. In this start position there is a mass point on the horizontal line at 3 o'clock if $\beta 0=\text{asin}(a/R)$

Tabulate $Z(\alpha)$ in intervals of one mth. of a θ degrees, $m := 1$

Each spoke rotates through the same angle α

$$j := 0, 1 .. m-1 \quad h_j := \frac{\theta}{m} \cdot j$$

In the above 'h' measures the rotation of each spoke, not each mass.

Put the values of $Z(\alpha)$ in the array 'ans':

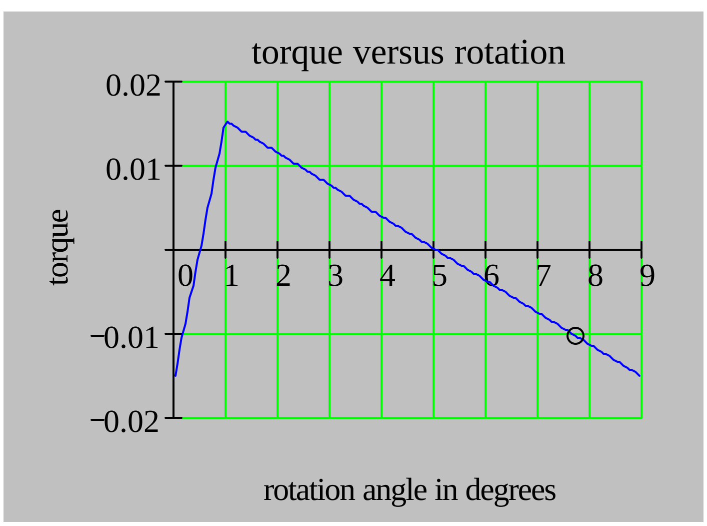
$$ans_j := Ztotal(h_j)$$

Plot of Torque versus α

$r := 154$

The black circle below marks the torque at r degrees

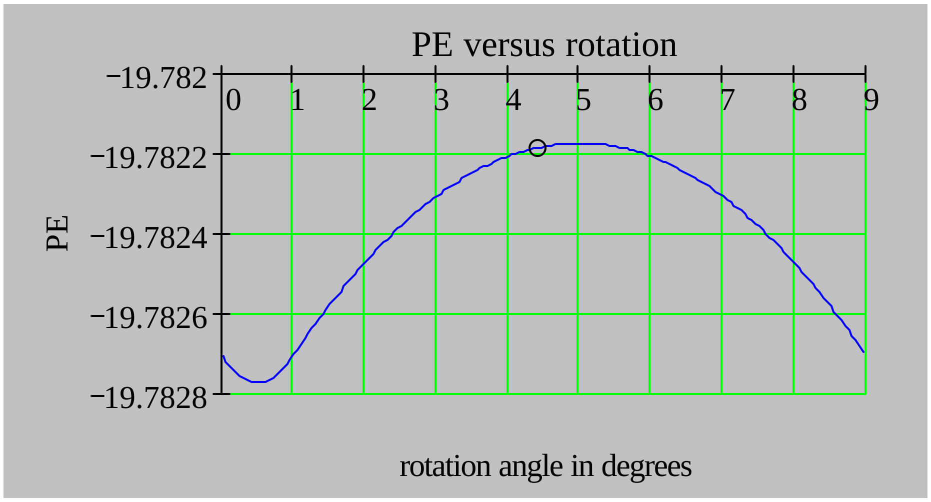
$m = 18$



Note that the torque vanishes in two places.

Plot of Potential Energy versus α

$r := 88$ in units of one mth. of a θ degrees $m = 180$



Variation of the torque and potential energy as the wheel is rotated anticlockwise through θ = 9º:

Plot the locus of Z(α) in the complex plane as α moves through 9º

$j := 0, 1 .. \text{rows}(ans) - 1$

$\text{potential}_j := \text{Im}(ans_j)$ $\text{torque}_j := \text{Re}(ans_j)$

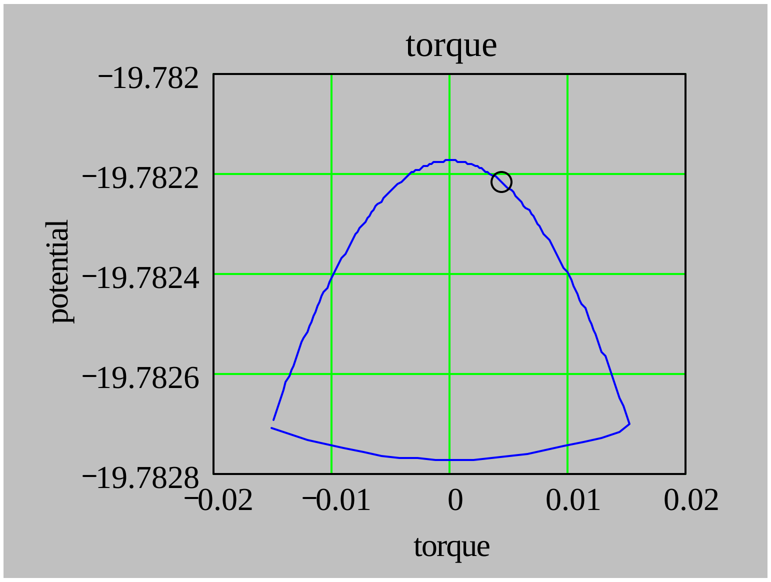
increase the value of r (0<r<179) to rotate anticlockwise from the start position in 1/20 degree steps

the value r = x corresponds to x/20 degrees anticlockwise:

change r below:

$r := 77$

$$\text{rotation} := \frac{r}{m} \cdot \frac{\theta}{\pi} \cdot 180 \qquad \text{rotation} = 3.85 \quad \text{deg}$$



T.J. Fairclough, October 1999

$r := 50$

$$\frac{\theta}{2}\Big) - r\cdot\cos\Big(\alpha + \chi - \frac{\theta}{2}\Big)\Big)$$

$$+ \chi - \frac{\theta}{2}\Big)\Big)$$

$$\frac{\theta}{2}\Big) - r\cdot\cos\Big(\alpha + \chi - \frac{3\cdot\theta}{2}\Big)\Big)$$

$$+ \chi - \frac{3\cdot\theta}{2}\Big)\Big)$$

$) + \mathrm{sum4}(n1, n2, \alpha)$

leg

deg

5 deg

6 deg

)]

ıths. of θ

80

mths. of θ

30